Exam Numerieke Wiskunde 1
August 24, 2012

Use of a simple calculator is allowed. All answers need to be motivated.
In front of the exercises you find its weight. In fact it gives the number of tenths which can be gained in the final mark. In total 6 points can be scored with this exam.

1. (a) [3] Sketch the function $f(x) = \exp(2x) + x$ on the interval [-1,0]. Next sketch the approximation the Newton method uses in its first step if it starts from $x_0 = 0$. Also give the equation for x_1.

   (b) [3] Consider the fixed point method $x_{n+1} = g(x_n)$ with fixed point p. Using the Taylor series show that for x_n close to p it holds that

   $$x_{n+1} - p = g'(p)(x_n - p) + \frac{1}{2}g''(p)(x_n - p)^2 + \ldots$$

   When will we have linear convergence and when quadratic?

   (c) [3] Express Newton's method in fixed point notation and show that for this case $g'(p) = 0$.

2. Suppose $f_1 = x_1 + x_2 - 1, \quad f_2 = \frac{1}{100} - \ln(1 + x_2 - x_1)$. We want to solve $f_1(x_1, x_2) = 0, \ f_2(x_1, x_2) = 0$, which is written in vector form as $\mathbf{f}(\mathbf{x}) = \mathbf{0}$. We want to solve it with a fixed point method $\mathbf{x_{n+1}} = \mathbf{g}(\mathbf{x_n})$ where $\mathbf{g}(\mathbf{x}) = \mathbf{x} + A\mathbf{f}(\mathbf{x})$.

   (a) [3] Give the Jacobian matrix of $\mathbf{f}$.

   (b) [2] Why is $(1/2, 1/2)$ a reasonable guess of the zero?

   (c) [4] Give the equation from which the best matrix A, based on the guess in the previous part, must be solved.

   (d) [3] The Jacobian of $\mathbf{g}$ at the fixed point using the A of the previous part is

   $$0.00502508 \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

   Why is this matrix relevant for the study of the convergence of the fixed point method? Will the method converge in the neighborhood of the fixed point and why?

3. (a) [3] Derive the quadratic interpolation polynomial for $f(x) = \exp(x)$ using Lagrange basis functions on three interpolation points 0, 1 and 2.

   (b) [2] Give the general form of the interpolation error when one uses $(n+1)$ interpolation points and determine with this an upperbound for the interpolation error on the interval [0,2] for interpolation in the previous part (given is that $|x(x-1)(x-2)| \le \frac{2}{9}\sqrt{3}$ on [0,2]).

   (c) [2] On an equidistant grid one can encounter the Runge phenomenon. Explain what this is. How does one cope with this problem? (In fact two solutions are possible. Discussing one of these is enough).

Continue on other side!

4. (a) $\boxed{2}$ Give the midpoint rule for integration on the interval $[a, b]$. Give also the interpolation polynomial on which this rule is based.

   (b) $\boxed{3}$ The trapezium rule is based on two interpolation points and is exact for linear polynomials. However, the midpoint rule is also exact for linear polynomials. Explain from a sketch why this happens.

   (c) $\boxed{3}$ Suppose we have a numerical method which approximates I by $I(h)$, where h is the mesh size and $I = I(h) + ch^4 + O(h^5)$ for some nonzero c. Derive a combination of $I(h)$ and $I(2h)$ that approximates I and which has error $O(h^5)$.

   (d) $\boxed{2}$ Describe the Romberg method.

5. (a) $\boxed{3}$ Describe the power method to find the largest eigenvalue of a matrix. Also show that the convergence depends on $|\lambda_2/\lambda_1|$ where λ_1 is the biggest eigenvalue and λ_2 the one but biggest eigenvalue and not equal in magnitude to λ_1.

   (b) $\boxed{3}$ Describe the Jacobi iteration method for solving $Ax = b$. Suppose that A is strictly diagonally dominant. Show that the Jacobi method converges for such matrices.

   (c) $\boxed{2}$ Explain why in linearly converging methods for $Ax = b$ the stopping criterion $||x^{(m+1)} - x^{(m)}||$ is small does not necessarily mean that $||x^{(m+1)} - x||$ is small, where x is the exact solution of $Ax = b$. When in particular is this the case?

   (d) $\boxed{2}$ Explain the idea of the gradient method and argue why in this method $J(x)$ is monotonically decreasing.

6. Consider on [0, 1] for $u(x, t)$ the convection equation $\partial u/\partial t = -\partial u/\partial x$ with initial condition $u(x, 0) = \sin(\pi x)$ and boundary condition $u(0, t) = \sin^2(t)$. Let the grid in x direction be given by $x_i = i\Delta x$ where $\Delta x = 1/m$.

   (a) $\boxed{3}$ Show that $u_x(x_i, t) = \frac{u(x_i,t)-u(x_{i-1},t)}{\Delta x} + O(\Delta x)$.

   (b) $\boxed{3}$ Show that the system of ordinary differential equations (ODEs) that results from using the expression in (a) is of the form

$$\frac{d}{dt}\vec{u}(t) = -\frac{1}{\Delta x}(I - B)\vec{u}(t) + \vec{b}(t)$$

   and give B and $\vec{b}(t)$.

   (c) $\boxed{3}$ Show that any eigenvalue of B will be at most one in magnitude. Sketch in the complex plane where the eigenvalues of $I - B$ are located.

   (d) $\boxed{3}$ Show that the numerical integration with the Forward/Explicit Euler method of the system of ODEs will be stable if $\Delta t/\Delta x \leq 1$.

Total $\boxed{60}$

Formula sheet Numerical Mathematics 1, 2012

Th. F.1 Mean value theorem If $f(x) \in C^\infty[a,b]$:

$$\exists \zeta \in (a,b) \text{ for which } f(b) - f(a) = f'(\zeta)(b-a)$$

Th. F.2 Mean value theorem in multidimensions Let $\vec{f}(\vec{x})$ be one times differentiable in all its components then

$$\vec{f}(\vec{b}) - \vec{f}(\vec{a}) = [\int_0^1 J_f(\vec{a} + t(\vec{b} - \vec{a}))dt](\vec{b} - \vec{a}).$$

where J_f is the Jacobian of $\vec{f}$.

Th. F.3 Weighted Mean value theorem for integrals If $f(x) \in C[a,b]$, and the sign of $w(x)$ is constant on $[a,b]$ and $w(x)$ integrable on $[a,b]$:

$$\exists \zeta \in (a,b) \text{ for which } \int_a^b w(x)f(x)dx = f(\zeta)\int_a^b w(x)dx$$

Def. F.4 Spectrum The spectrum is the set of all eigenvalues of a matrix A and denoted by $\sigma(A)$

Def. F.5 Spectral radius The spectral radius $\rho(A) = \max_{\lambda \in \sigma(A)} |\lambda|$. Alternative naming $r_\sigma(A)$.

Def. F.6 p-norm The p-norm for vectors $x \in C^n$ $(p > 0)$ is given by

$$||x||_p = \{\sum_{i=1}^n |x_i|^p\}^{\frac{1}{p}}$$

Special cases

$$\begin{aligned} ||x||_1 &= \sum_{i=1}^n |x_i| \text{ one norm} \\ ||x||_2 &= \sqrt{(x,x)} \text{ two norm or Euclidian norm} \\ ||x||_\infty &= \max_i |x_i| \text{ infinity norm or maximum norm} \end{aligned}$$

Def. F.7 Induced matrix norm Vector p-norms induce matrix p-norms by

$$||A||_p = \max_x \frac{||Ax||_p}{||x||_p} = \max_{x, ||x||_p = 1} ||Ax||_p$$

For the special vector norms in F.6 this leads to

$$\begin{aligned} ||A||_1 &= \max_j \sum_{i=1}^n |A_{ij}|, \text{ one norm or column norm} \\ ||A||_2 &= \sqrt{\rho(A^*A)}, \text{ two norm or spectral norm} \\ ||A||_\infty &= \max_i \sum_{j=1}^n |A_{ij}| = ||A^T||_1, \ \infty\text{-norm, max norm or row norm} \end{aligned}$$

Th. F.8 For all $p > 0$, $\rho(A) \leq ||A||_p$.
Def. F.9 Similarity Two matrices A and B are called similar if there exists a non-singular matrix Q such that $B = Q^{-1}AQ$.
Def. F.10 Diagonalizability A matrix A is called diagonalizable if it is similar to a diagonal matrix.
Def. F.11 Jordan block A Jordan block is a square matrix of the form

$$B(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ 0 & \cdots & & 0 & \lambda & 1 \\ 0 & \cdots & & 0 & 0 & \lambda \end{bmatrix}$$

Th. F.12 A matrix consisting of one Jordan block has only one eigenvector.
Th. F.13 Jordan normal form Any matrix A is similar to a block diagonal matrix J with Jordan blocks $B(\lambda_i)$ on the diagonal with λ_i running over the eigenvalues of A.
Th. F.14 For a diagonalizable matrix the size of the Jordan blocks are all 1. Moreover it has a complete set of eigenvectors, meaning that if the order of the matrix is N then the eigenvectors span the space C^N.
Th. F.15 $(B(\lambda))^m$ goes to zero for $m \to \infty$ only and only if $|\lambda| < 1$. Moreover, this is the case if we replace the 1's on the superdiagonal by arbitrary numbers (they do not need to be equal).
Def. F.16 Positive Definite A complex matrix A (not necessarily Hermitian) is positive definite if for all complex vectors x, $(x, Ax) > 0$ for $x \neq 0$.
Th. F.17 If λ is an eigenvalue of a real matrix then also $\bar{\lambda}$ is.
Th. F.18 A real symmetric matrix has only real eigenvalues.
Th. F.19 A real symmetric positive definite matrix has only positive real eigenvalues.
Def F.20 Strict diagonal dominance A matrix A of order N is called stictly diagonally dominant when

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{N} |a_{ij}| \quad \text{for} \quad i = 1, 2, \ldots, N$$